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Quantised gauge theory, dimensional reduction and OSp(4 + N/2) supersymmetry

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Abstract. The Klein–Kaluza version of the combined Einstein–Yang–Mills system is based on a larger-dimension manifold, admitting an O($4+N$) symmetry, which undergoes compactification. The theory provides a unified treatment of all gauge fields and its action consists of a single invariant Lagrangian and, separately, a single gauge-fixing term. We show that both Lagrangians can be combined within a larger framework, admitting an OSp($4+N/2$) supersymmetry, which undergoes appropriate dimensional reduction; here the extra graded dimensions are associated with the fictitious fields needed for consistent quantisation and unitarity.

1. Introduction

We are becoming accustomed to viewing four-dimensional general relativistic gauge theories as appropriate compactified versions of higher-dimensional theories (Cremmer 1981). This idea possesses an old pedigree (Kaluza 1921, Klein 1926, DeWitt 1965, Cho 1975) but it has remained dormant a while. Its recent revival is largely due to the impetus received from developments in supergravity (Van Nieuwenhuizen 1981).

A quantised gauge theory must of course include the fictitious (ghost) fields as well as the original gauge field.. In a recent publication, Hosoya *et al* (1981) have demonstrated that the ghost Lagrangian, like the starting classical Lagrangian, can be incorporated very naturally within the $(4+N)$-dimensional Kaluza–Klein framework. Specifically they show that the full Lagrangian

$$\mathscr{L} = K^{-2}\sqrt{-\hat{g}}\;\hat{R} + K^{-1}\delta(\partial_{\hat{\mu}}\bar{\omega}_{\hat{\nu}}\sqrt{-\hat{g}}\,g^{\hat{\mu}\hat{\nu}}) \tag{1}$$

is nothing else but the usual gauge plus fictitious Lagrangians when decomposed into more familiar components of the generalised metric $\hat{g}$, associated with the compactified $(4+N)$-dimensional manifold. In (1) the indices $\hat{\mu}$, $\hat{\nu}$ run from 1 to $4+N$, $\hat{R}$ is the full curvature, $\bar{\omega}$ stands for the dual set of ghost fields and δ represents the ordinary Becchi–Rouet–Stora variation of the collection, as spelt out in their paper.

In parallel with these developments has come the realisation that gauge fields and ghost fields can be accommodated within a larger framework which includes space–time plus two (scalar) graded dimensions. The basic scheme admits an OSp(4/2) supersymmetry and it leads very naturally to an extended set of BRS invariances for a wide class of Lagrangians. This happens both for Yang–Mills theory (Delbourgo and Jarvis 1982, Bonora and Tonin 1981) and, in a local version, for gravity too (Delbourgo *et al* 1982a, b, Pasti and Tonin 1982).

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In this paper we wish to combine both ideas and examine the consequences of assuming a $(4+N+2)$-dimensional manifold, admitting an $\mathrm{OSp}(4+N/2)$ symmetry, so that an extended version of (1) emerges in which the ghost fields are treated symmetrically as part of a multiplet structure which includes the gravitational and internal gauge fields. We shall show that such a generalisation can be successfully accomplished and demonstrate how it leads to an extended set of BRS invariances for the combined system, with corresponding extended gauge identities. From our viewpoint the quantised, generally covariant gauge theory is regarded as a dimensionally reduced version of a $4+N+2$ space–time theory, where the last two dimensions are graded; we believe that the result is a new and *useful* application of supersymmetry to the real world.

2. Extended BRS variations

For all its conceptual simplicity, the algebraic complexity of the formalism demands that we introduce a notation that is easily intelligible. We shall let $X^M=(x^\mu, \xi^{\mathfrak{m}}, \theta^m)$ denote the coordinates of the full manifold; x^μ refers to ordinary space–time, $\xi^{\mathfrak{m}}$ to the internal space and θ^m to the graded dimensions. Thus μ runs from 0 to 3, $\mathfrak{m}$ from 1 to N and $m=1, 2$. Tangent space indices are denoted by early letters of the alphabet, $A=(\alpha, \mathfrak{a}, a)$. Some formulae require sign factors $[MN]$, $[AB]$ which are $+1$ except for $[mn]=-1$, $[ab]=-1$, and so on.

The basic idea is to introduce a generalised metric G transforming in the normal manner

$$G^{MN}(X)=\frac{\partial X^M}{\partial X_0^R}\frac{\partial X^N}{\partial X_0^S}\mathring{G}^{SR}(X_0)[RN] \tag{2}$$

or better still the vielbein,

$$E_A^M(X)=(\partial X^M/\partial X_0^N)\mathring{E}_A^N(X_0), \qquad G^{MN}=E_A^M E_B^N \eta^{AB}[NA]. \tag{3}$$

In line with earlier work we shall suppose that at the point X_0 the metric or vielbein depends in a factorisable way upon x_0 and ξ_0 (just as in Kaluza–Klein theory) and that the off-diagonal graded components vanish. Specifically, at X_0, we take

$$E_A^M=\begin{pmatrix} e_\alpha^\mu, & -e_\alpha^\rho A_\rho^{\mathfrak{m}}, & 0 \\ 0, & e_{\mathfrak{a}}^{\mathfrak{m}}, & 0 \\ 0, & 0, & \delta_a^m \end{pmatrix} \tag{4}$$

or

$$G^{MN}=\begin{pmatrix} g^{\mu\nu}, & -g^{\mu\rho}A_\rho^{\mathfrak{m}}, & 0 \\ -g^{\rho\nu}A_\rho^{\mathfrak{n}}, & g^{\mathfrak{mn}}+g^{\rho\sigma}A_\rho^{\mathfrak{m}}A_\sigma^{\mathfrak{n}}, & 0 \\ 0, & 0, & \varepsilon^{mn} \end{pmatrix} \tag{5}$$

where

$$A_\rho^{\mathfrak{m}}(x_0, \xi_0)=e_{\mathfrak{a}}^{\mathfrak{m}}(\xi_0)A_\rho^{\mathfrak{a}}(x_0). \tag{6}$$

In (4) e_α^μ is the usual gravitational vierbein, while $e_{\mathfrak{a}}^{\mathfrak{m}}$ in (6) is the internal space vielbein or 'velocity field', satisfying $\delta^{\mathfrak{ab}}e_{\mathfrak{a}}^{\mathfrak{m}}(\xi)e_{\mathfrak{b}}^{\mathfrak{n}}(\xi)=g^{\mathfrak{mn}}(\xi)$, and the Lie algebra commutation rule

$$e_{\mathfrak{a}}^{\mathfrak{m}}\partial_{\mathfrak{m}}e_{\mathfrak{b}}^{\mathfrak{n}}-e_{\mathfrak{b}}^{\mathfrak{m}}\partial_{\mathfrak{m}}e_{\mathfrak{a}}^{\mathfrak{n}}=f_{\mathfrak{ab}}{}^{\mathfrak{c}}e_{\mathfrak{c}}^{\mathfrak{n}}, \qquad \partial_{\mathfrak{m}}\equiv\partial/\partial\xi^{\mathfrak{m}}. \tag{7}$$

For further details see Hosoya *et al* (1981). The point about the special choice (4) is that it has *vanishing curvature along the θ-directions.*

As in earlier work, we now perform a special θ-dependent transformation which guarantees flatness along the graded dimensions still, namely

$$x^\mu = x_0^\mu + \theta_0^m \omega_m^\mu(x_0) + \tfrac{1}{2}\theta_0^2 B^\mu(x_0), \qquad \xi^{\mathscr{m}} = \xi_0^{\mathscr{m}} + \theta_0^m \omega_m^{\mathscr{m}}(x_0, \xi_0) + \tfrac{1}{2}\theta_0^2 B^{\mathscr{m}}(x_0, \xi_0), \tag{8}$$
$$\theta^m = \theta_0^m, \qquad \theta^2 \equiv \varepsilon^{mn}\theta_m\theta_n$$

and follow this by a supertranslation, $\theta^m \to \theta^m + \varepsilon^m$. Comparing the result with a coordinate transformation

$$x^\mu \to x^\mu - \varepsilon^m \omega_m^\mu, \qquad \xi^{\mathscr{m}} \to \xi^{\mathscr{m}} - \varepsilon^m \omega_m^{\mathscr{m}}, \qquad \theta^m \to \theta^m + \varepsilon^m,$$

followed by a special transformation (8) at the new point X, we are able to discover the extended BRS variations of the ghost fields ω and auxiliary fields B. These read

$$\delta\omega_m^\mu = \varepsilon_n \omega^{n\nu}\partial_\nu \omega_m^\mu + \varepsilon_m B^\mu, \qquad \delta B^\mu = \varepsilon_n \omega^{n\nu}\partial_\nu B^\mu, \tag{9a}$$

$$\delta\omega_m^{\mathscr{m}} = \varepsilon_n \omega^{n\nu}\partial_\nu \omega_m^{\mathscr{m}} + \varepsilon_n \omega^{n\mathscr{n}}\partial_{\mathscr{n}} \omega_m^{\mathscr{m}} + \varepsilon_m B^{\mathscr{m}}, \qquad \delta B^{\mathscr{m}} = \varepsilon_n \omega^{n\nu}\partial_\nu B^{\mathscr{m}} + \varepsilon_n \omega^{n\mathscr{n}}\partial_{\mathscr{n}} B^{\mathscr{m}}. \tag{9b}$$

The set $(9a)$ is easily recognised as the extended gravitational BRS transformations. On the other hand, the set $(9b)$ are the extended BRS variations for the internal gauge fields, but in disguise. We strip off this disguise by defining tangent space fields as follows†:

$$\begin{aligned} \omega_m^{\mathscr{m}}(x, \xi) &= \omega_m^{\mathscr{a}}(x) e_{\mathscr{a}}^{\mathscr{m}}(\xi), \\ B^{\mathscr{m}}(x, \xi) &= B^{\mathscr{a}}(x) e_{\mathscr{a}}^{\mathscr{m}}(\xi) - \tfrac{1}{2}\omega^{\mathscr{a}m}(x)\omega_m^{\mathscr{b}}(x) e_{\mathscr{b}}^{\mathscr{n}}(\xi)\partial_{\mathscr{n}} e_{\mathscr{a}}^{\mathscr{m}}(\xi). \end{aligned} \tag{9c}$$

In terms of the new fields,

$$\begin{aligned} \delta\omega_m^{\mathscr{a}} &= \varepsilon_m B^{\mathscr{a}} + \varepsilon_n \omega^{n\nu}\partial_\nu \omega_m^{\mathscr{a}} + \tfrac{1}{2} f_{\mathscr{b}\mathscr{c}}{}^{\mathscr{a}} \varepsilon_n \omega^{n\mathscr{b}} \omega_m^{\mathscr{c}}, \\ \delta B^{\mathscr{a}} &= \varepsilon_n \omega^{n\nu}\partial_\nu B^{\mathscr{a}} + \tfrac{1}{2} f_{\mathscr{b}\mathscr{c}}{}^{\mathscr{a}} \varepsilon_n \omega^{n\mathscr{b}} B^{\mathscr{c}} + \tfrac{1}{12} f_{\mathscr{b}\mathscr{e}}{}^{\mathscr{d}} f_{\mathscr{d}\mathscr{c}}{}^{\mathscr{a}} \varepsilon_n \omega^{n\mathscr{b}} \omega_m^{\mathscr{e}} \omega^{\mathscr{c}m}, \end{aligned} \tag{9d}$$

are more easily identified as the combined gravitational–internal transformations (Delbourgo and Jarvis 1982, Delbourgo *et al* 1982b)‡. We may also derive the extended transformations of the metric from the rule

$$G^{MN}(x, \xi, \varepsilon) = G^{MN}(x, \xi, 0) + \varepsilon^m \partial_m G^{MN}(x, \xi, 0) + \ldots$$

consistently with (9). In this way we derive

$$\begin{aligned} \delta g^{\mu\nu} &= \varepsilon^m (g^{\lambda\nu}\partial_\lambda \omega_m^\mu + g^{\mu\lambda}\partial_\lambda \omega_m^\nu - \omega_m^\lambda \partial_\lambda g^{\mu\nu}), \\ \delta g^{\mathscr{m}\mathscr{n}} &= \varepsilon^m (g^{\ell\mathscr{n}}\partial_\ell \omega_m^{\mathscr{n}} + g^{\mathscr{m}\ell}\partial_\ell \omega_m^{\mathscr{n}} - \omega_m^\ell \partial_\ell g^{\mathscr{m}\mathscr{n}}), \end{aligned} \tag{9e}$$

and the off-diagonal variation (see (5)), which simplifies to

$$\delta A_\mu^{\mathscr{a}} = \varepsilon_m (\partial_\mu \omega^{\mathscr{a}m} + \omega^{\lambda m}\partial_\lambda A_\mu^{\mathscr{a}} + A_\lambda^{\mathscr{a}}\partial_\mu \omega^{\lambda m} + f_{\mathscr{b}\mathscr{c}}{}^{\mathscr{a}} \omega^{\mathscr{b}m} A_\mu^{\mathscr{c}}), \tag{9f}$$

thereby completing the extended BRS set.

† The origin of the terms in $(9c)$ is not so mysterious when we realise that the full right group action is $R\exp(\lambda T)\xi^{\mathscr{m}} = \xi^{\mathscr{m}} + \lambda^{\mathscr{a}} e_{\mathscr{a}}^{\mathscr{m}} + \tfrac{1}{2}\lambda^{\mathscr{a}}\lambda^{\mathscr{b}} e_{\mathscr{a}}^{\ell}\partial_\ell e_{\mathscr{b}}^{\mathscr{m}} + \ldots$. Equation $(9c)$ expresses the fact that the internal part of (8) is of this form, with $\lambda T \equiv \theta_0^m \omega_m^{\mathscr{a}} T^{\mathscr{a}} + \tfrac{1}{2}\theta_0^2 B^{\mathscr{a}} T^{\mathscr{a}}$ (cf Delbourgo and Jarvis 1982).

‡ It should be noted that in (9) both ghosts and the multiplier field B enter on an equal footing; this is to be compared with Hosoya *et al*'s (1981) asymmetric treatment of the ghost ω and that of the dual ghost and auxiliary field.

3. Actions

When it comes to constructing BRS invariant actions we can take our cue from earlier research. One already knows (Cho 1975) that the full $(4+N)$-dimensional curvature R breaks up into that for gravity, the Yang–Mills term F^2, plus a (constant) internal space contribution. It is just a matter of recasting this into superfield form, which is no problem. Thus

$$S_{\text{gauge}} = (VK^2)^{-1} \int d^{N+6}X \sqrt{-G} X^2 R^{MN} G_{NM} \tag{10}$$

precisely decomposes into its conventional parts. In (10) the measure is

$$\sqrt{-G}\, d^{N+6}X = \sqrt{-g}\, d^4x \sqrt{\mathscr{g}}\, d^N\xi\, d^2\theta, \qquad V = \int \sqrt{\mathscr{g}}\, d^N\xi,$$

and the X^2 factor is there to cancel out the θ integration—it does not spoil the supertranslation invariance of the action. However a wide variety of Lorentz-covariant gauge-fixing Lagrangians is possible, including†

$$\begin{array}{lll} E^\mu_\alpha E^\nu_\beta \eta^{\alpha\beta}\eta_{\mu\nu}, & E^\mu_a E^\nu_\ell \delta^{a\ell}\eta_{\mu\nu}, & E^\mu_a E^\nu_b \varepsilon^{ab}\eta_{\mu\nu}, \\ E^m_\alpha E^n_\beta \eta^{\alpha\beta} g_{mn}, & E^m_a E^n_\ell \delta^{a\ell} g_{mn}, & E^m_a E^n_b \varepsilon^{ab} g_{mn}, \\ E^m_\alpha E^n_\beta \eta^{\alpha\beta}\varepsilon_{ab}, & E^m_a E^n_\ell \delta^{a\ell}\varepsilon_{mn}, & E^m_a E^n_b \varepsilon^{ab}\varepsilon_{mn}. \end{array} \tag{11}$$

All of these of course break the general and internal gauge covariance, as they must. Remembering the special forms (4) and the class of transformations (8) taking us from X_0 to X, it turns out that θ-flatness always entails

$$E^a_\alpha = E^a_a = E^\mu_a = 0, \qquad E^m_a = \delta^m_a, \qquad E^m_a = e^m_a(\xi).$$

Hence, apart from constants, we are left with four possibilities for the action

$$S_{\text{fixing}} = \int d^{N+6}X \sqrt{-G}(\gamma^{-1}E^\mu_\alpha E^\nu_\beta \eta^{\alpha\beta}\eta_{\mu\nu} + \gamma'^{-1}E^\mu_a E^\nu_b \varepsilon^{ab}\eta_{\mu\nu} + \zeta^{-1}E^m_\alpha E^n_\beta \eta^{\alpha\beta} g_{mn} + \zeta'^{-1}E^m_a E^n_b \varepsilon^{ab} g_{mn}) \tag{12}$$

among the class of Lorentz-covariant gauges. It is possible to rescale the fields B and ω so that $\gamma = \gamma'$ and $\zeta = \zeta'$, leaving us with two gauge parameters: γ specifying the gravitational gauge and ζ the internal space gauge. Again, up to an irrelevant constant, we can rewrite

$$S_{\text{fixing}} = V^{-1} \int d^{N+6}X \sqrt{-G}(G^{\mu\nu}\eta_{\mu\nu}\gamma^{-1} + G^{mn} g_{mn}\zeta^{-1}) \tag{13}$$

in its most elegant form‡. It turns out that for $\gamma \to 0$ the Lagrangian (Nakanishi 1978) admits a larger choral symmetry (Nakanishi 1980) but we need not restrict ourselves to that particular value.

It goes without saying that (13) possesses the extended BRS invariance under the set (9); the Lagrangian merely changes by a pure divergence under the BRS transformations. To make contact with the traditional fields we evaluate $G^{\mu\nu}$ and G^{mn} in terms

† The reason why we take g_{mn} rather than δ_{mn} is because that is the proper form-invariant metric, in the same sense that $\eta_{\mu\nu}$ is the Lorentz-invariant metric.

‡ $(\omega^m \times \omega_m)^a \equiv \varepsilon^{mn}\omega^\ell_m \omega^c_n f^a_{\ell c}$, etc.

of the component gauge fields (5) using the transformation (8) and changing to the location ξ via

$$\xi_0^{m} = \xi^{m} - \theta^m \omega_m^{m}(x_0, \xi) - \tfrac{1}{2}\theta^2[\omega^{n\kappa}\partial_\kappa \omega_m^{m}(x_0, \xi) + B^{m}(x_0, \xi)].$$

The measure being a group invariant, we can determine (13) at x_0 and the general internal location ξ. The result is finally†

$$S_{\text{fixing}} = \int d^4x\, \sqrt{-g}[\![\gamma^{-1}\eta_{\mu\nu}(g^{\mu\lambda}\partial_\lambda B^\nu + g^{\nu\lambda}\partial_\lambda B^\mu + g^{\kappa\lambda}\partial_\kappa \omega^{\mu m}\partial_\lambda \omega_m^\nu + 2B^\mu B^\nu)$$
$$+\zeta^{-1}(g^{\mu\nu}[-A_\mu^a\partial_\nu B^a + \partial_\mu \omega_m^a \partial_\nu \omega^{am} + \tfrac{1}{2}eA_\mu^a(\omega^m \times \partial_\nu \omega_m)^a]$$
$$+B^aB^a + \tfrac{1}{4}(\omega^m \times \omega_m)^a(\omega^n \times \omega_n)^a)]\!]. \tag{14}$$

Generalisations of (13) and (14) are possible wherein the density is weighted by the factor $(\sqrt{-G})^p$; this will not disturb the extended BRS invariance. Also one could envisage classes of axial gauges which take us outside the Lorentz-invariant set. Our methods will adequately cover these cases too. We have no reason to suppose that the extended gauge identities arising from the variations (9) will not be satisfied, because a regularisation procedure (4 dimensions $\rightarrow 2l$ dimensions) exists that always respects them.

In conclusion, we have shown that a generally covariant quantised gauge theory—including the fictitious fields—can be considerably viewed as a dimensionally reduced theory in $4+N+2$ dimensions in the Klein–Kaluza sense, where the last two dimensions are graded. It could well be that higher terms in normal mode expansions of the $(N+6)$-dimensional theory will lead to several more massive excitations (Salam and Strathdee 1981) than originally conceived, namely those of the ghosts.

Note added in proof. Ohkuwa, in a recent Osaka publication, arrives at our results by a different route.

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† $\gamma = \zeta = 0$ will be seen to yield the harmonic-Landau gauges; $\gamma = \zeta = 1$ give instead the de Donder–Fermi gauge. Obviously there is no obligation to choose the gauge parameters equal.